RELATIONS BETWEEN THE COMPONENTS OF THE CORRELATION FUNCTIONS OF AN ELASTIC FIELD

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Equilibrium and compatibility equations are used to obtain differential equations connecting algebraically independent components of the binary tensor correlation functions of a statistically homogeneous and isotropic elastic field. The cases of a birotational field for the stress tensor and of a potential field for the deformation tensor are investigated. Also obtained are the corresponding relations for the correlation tensors connecting the fields of curvature and the angle of rotation vectors. In the latter case the resulting equation is formally identical with the Kármán relation in the statistical hydrodynamics.

1. The stress and strain correlation functions are tensors of the fourth rank. Unlike the fourth rank tensor of the elastic moduli which has two independent components in the isotropic medium, the fourth rank correlation tensor is determined by five algebraic functions. The necessity of introducing five different functions follows from the axial symmetry of the problem, as an isolated axis appears in an isotropic medium, passing through the points between which the correlations are established.

We shall consider a statistically homogeneous medium. The binary correlation functions of the stress σ_{ij} and strain ε_{ij} tensor of such a medium are given by

$$S_{ijkl}(\mathbf{r}) \equiv \langle \sigma_{ij}^\circ(\mathbf{r}+\mathbf{r}_1)\,\sigma_{kl}^\circ(\mathbf{r}_1)\rangle, \qquad E_{ijkl}(\mathbf{r}) = \langle \varepsilon_{ij}^\circ(\mathbf{r}+\mathbf{r}_1)\,\varepsilon_{kl}^\circ(\mathbf{r}_1)\rangle \tag{1.1}$$

$$\sigma_{ij}^\circ(\mathbf{r}) \equiv \sigma_{ij}(\mathbf{r}) - \langle \sigma_{ij}(\mathbf{r})\rangle, \qquad \varepsilon_{ij}^\circ(\mathbf{r}) \equiv \varepsilon_{ij}(\mathbf{r}) - \langle \varepsilon_{ij}(\mathbf{r})\rangle \tag{1.2}$$

Here the angular brackets $\langle\ \rangle$ denote the statistical mean, and the superscript ($^\circ$) denotes the random components of the corresponding quantities.

For a statistically isotropic medium the stress and strain correlation tensors can be written in the form [1]

$$S_{ijkl}(\mathbf{r}) = S_\alpha(r)\,J_{ijkl}^\alpha, \qquad E_{ijkl}(\mathbf{r}) = E_\alpha(r)\,J_{ijkl}^\alpha, \qquad \alpha = 1,\ldots,5 \tag{1.3}$$

$$J_{ijkl}^1 = \delta_{ij}\delta_{kl}, \quad J_{ijkl}^2 = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}, \quad J_{ijkl}^3 = \delta_{ij}n_{kl} + \delta_{kl}n_{ij}$$
$$J_{ijkl}^4 = \delta_{ik}n_{jl} + \delta_{jl}n_{ik} + \delta_{il}n_{jk} + \delta_{jk}n_{il}, \qquad J_{ijkl}^5 = n_{ijkl} \tag{1.4}$$
$$n_{ij\cdot\cdot m} = n_i n_j \cdot\cdot n_m, \qquad n_i = x_i / r$$

where the twice occurring indices denote summation and the coordinates of the points between which the correlation is established, are 0 and x_i.

The expressions (1.3) and (1.4) show that when $r \to 0$ and $\alpha = 3,4,5$, the functions $S_\alpha(r)$ and $E_\alpha(r)$ vanish. At the other extreme, when the distances become large, all components of the stress and strain correlation tensors decrease asymptotically provided that no long-range order exists in the distribution of inhomogeneities.

2. The relations connecting the components of the correlation tensor show a significant dependence on the type of the field under investigation, whether it is potential or birotational. Kroner [2] has shown that any second rank tensor field $\mathbf{T}$ can be decomposed into a potential field $\mathbf{T}_1$ and a birotational field $\mathbf{T}_2$

$$\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2, \qquad \operatorname{Rot} \mathbf{T}_1 = 0, \qquad \operatorname{div} \mathbf{T}_2 = 0 \tag{2.1}$$

the potential field described by the vector potential and the birotational field – by the tensor potential

$$\mathbf{T}_1 = \operatorname{def} \varphi, \qquad \mathbf{T}_2 = \operatorname{Rot} \mathbf{\Phi}, \qquad \operatorname{div} \mathbf{\Phi} = 0 \tag{2.2}$$

$$\operatorname{def}_i \varphi_k \equiv \varphi_{(k,\, i)}, \qquad \operatorname{Rot}_{ijkl} \equiv e_{ink} e_{jml} \nabla_n \nabla_m \tag{2.3}$$

Here the index following the coma denotes differentiation with respect to the corresponding coordinate, the brackets denote symmetrization with respect to the indices contained therein and e_{ink} denotes the antisymmetric unit tensor.

Taking into account the relations

$$\varepsilon_{ik} = u_{(k,\, i)}, \qquad \sigma_{ij,\, j} = 0 \tag{2.4}$$

the first of which represents the condition that the deformations are small and the second is the equation of equilibrium with the volume forces absent; we find that the strain field is potential and that the displacement vector $\mathbf{u}$ can be regarded as the potential of this field. Conversely, the stress field is birotational in the absence of the volume forces.

It is for this reason that, although the tensor representations of the stress and strain correlation fields are identical, differences in the form of the sought relationships connecting the components of the corresponding tensors are to be expected.

3. We first find the relations connecting the components of the stress correlation tensor. We assume that the stress field has no potential components and is purely birotational. Then, assuming for definiteness that $n_1 = n_2 = 0$ and $n_3 = n$ and passing from the tensor to the matrix indices, we find, using the first equation of (1.3), that

$$\begin{aligned} S_{ijkl} = S_{12} J^1_{ijkl} + S_{66} J^2_{ijkl} + (S_{13} - S_{12}) J^3_{ijkl} + \\ + (S_{44} - S_{66}) J^4_{ijkl} + (S_{11} + S_{33} - 2S_{13} - 4S_{44}) J^5_{ijkl} \end{aligned} \tag{3.1}$$

Only five of the six components of the stress correlation tensor appearing in the right-hand side of (3.1) are algebraically independent, as the axial symmetry implies the relation

$$S_{66} = {}^1/_2 \, (S_{11} - S_{12}) \tag{3.2}$$

Using (1.1) and (2.4) we find that for a birotational field the condition

$$S_{ijkl,\, j} = 0 \tag{3.3}$$

must hold. Inserting (3.1) into (3.3) we obtain

$$A_1 n_i \delta_{kl} + A_2 (n_l \delta_{ik} + n_k \delta_{il}) + A_3 n_{ikl} = 0 \tag{3.4}$$

$$A_1 \equiv S_{13}' + \frac{2}{r} (S_{13} + S_{44} - S_{12} - S_{66}),$$

$$A_2 \equiv S_{44}' + \frac{1}{r} (3S_{44} + S_{13} - S_{12} - 3S_{66})$$

$$A_3 \equiv S_{33}' - S_{13}' - 2S_{44}' + \frac{2}{r} (2S_{11} + S_{33} - 3S_{13} - 6S_{44}) \tag{3.5}$$

Here the prime denotes the derivative with respect to the scalar argument. Assigning various values to the tensor indices in (3.4), we find that every $A_\varkappa$ must be equal to zero. This yields three differential equations connecting five algebraically independent components of the matrix S_{pq}

$$
\begin{aligned}
&rS_{13}' + 2(S_{13} + S_{44} - S_{12} - S_{66}) = 0\\
&rS_{33}' + 2(S_{33} - S_{13} - 2S_{44}) = 0\\
&rS_{44}' + 3S_{44} + S_{13} - S_{12} - 3S_{66} = 0
\end{aligned} \tag{3.6}
$$

Relations of the type (3.6) were first obtained in [1]. However an error was committed in numerical computations, which caused the omission of some terms from the final equations.

4. We shall use the first equation of (2.4) to obtain the relations connecting the components of the stress correlation tensor. This yields [3]

$$E_{ijkl} = -\nabla_{(i}U_{j)(k,l)}, \qquad U_{ij}(\mathbf{r}) \equiv \langle u_i^\circ(\mathbf{r}+\mathbf{r}_1)\, u_j^\circ(\mathbf{r}_1)\rangle \tag{4.1}$$

where U_{ij} is the correlation tensor of the displacement vectors. The latter can be written as

$$U_{ij} = U_1\delta_{ij} + U_2 n_{ij} \tag{4.2}$$

Inserting (4.2) into the first equation of (4.1) we obtain

$$
\begin{aligned}
r^2E_{ijkl} = &-U_2J^1_{ijkl} - {}^1/_2(U_2 + rU_1')J^2_{ijkl} + (2U_2 - rU_2')J^3_{ijkl} +\\
&+ {}^1/_4(6U_2 - 3rU_2' + rU_1' - r^2U_1'')J^4_{ijkl} - (8U_2 - 5rU_2' + r^2U_2'')J^5_{ijkl}
\end{aligned} \tag{4.3}
$$

On the other hand, we can express the stress correlation tensor in terms of its matrix components using the second equation of (1.3)

$$
\begin{aligned}
E_{ijkl} = &E_{12}J^1_{ijkl} + {}^1/_4E_{66}J^2_{ijkl} + (E_{13} - E_{12})J^3_{ijkl} +\\
&+ {}^1/_4(E_{44} - E_{66})J^4_{ijkl} + (E_{11} + E_{33} - 2E_{13} - E_{44})J^5_{ijkl}
\end{aligned} \tag{4.4}
$$

Here the passage from the tensor to the matrix notation involved numerical coefficients [4] which were brought in according to the rule $\varepsilon_4 = 2\varepsilon_{23}$ and $\varepsilon_6 = 2\varepsilon_{12}$, and instead of (3.2) we now have

$$E_{66} = 2(E_{11} - E_{12}) \tag{4.5}$$

Comparing (4.3) and (4.4) we obtain the following system of equations:

$$
\begin{aligned}
&r^2E_{12} = -U_2, \quad r^2E_{66} = -2(U_2 + rU_1'), \quad r^2(E_{13} - E_{12}) = 2U_2 - rU_2'\\
&r^2(E_{44} - E_{66}) = 6U_2 - 3rU_2' + rU_1' - r^2U_1''\\
&r^2(E_{11} + E_{33} - 2E_{13} - E_{44}) = -8U_2 + 5rU_2' + r^2U_2''
\end{aligned} \tag{4.6}
$$

The required relationships connecting the components of the correlation matrix of the strain field are now obtained by eliminating the auxilliary functions U_1 and U_2

$$
\begin{aligned}
&rE_{11}' + 2E_{11} - 3E_{12} + E_{13} - E_{44} = 0, \quad rE_{12}' + E_{12} - E_{13} = 0\\
&rE_{13}' - E_{11} + 3E_{12} - E_{13} - E_{33} + E_{44} = 0
\end{aligned} \tag{4.7}
$$

Relations (3.6) and (4.7) determine the connections between the components of the binary correlation tensors of the stress and strain fields. Comparing them we find that in the first case S_{13} and S_{33} should be chosen as the two independent components, and in the second case – E_{11} and E_{12}. The remaining components of the correlation tensors can then be expressed in terms of two base components using the differential operators. Performing the computations, we find

$$8S_{11} = (R+2)(R+4)S_{33}, \qquad 4S_{44} = (R+2)S_{33} - 2S_{13}$$
$$8S_{12} = 8(R+1)S_{13} - R(R+2)S_{33} \tag{4.8}$$

$$E_{33} = (R+1)E_{11} + R(R+1)E_{12}, \quad E_{44} = (R+2)E_{11} + (R-2)E_{12}$$
$$E_{13} = (R+1)E_{12}, \quad R \equiv r\frac{d}{dr} \tag{4.9}$$

The relations (3.6) and (4.7) obtained, or their equivalents (4.8) and (4.9), express the connections between the components of the correlation tensors of the purely birotational and purely potential field. If on the other hand the random second rank tensor field is mixed, the relations obtained cannot be utilized until it is decomposed with the help of (2.1) and (2.2) into its potential and birotational components.

5. When the stress and strain correlation tensors are computed, usually the angle of rotation correlation tensor is computed as well. The latter is particularly useful in estimating the relative disorientation of the grains during the deformation of microinhomogeneous media. The relationship connecting the components of the latter tensor can also be obtained as follows. Defining the rotation vector $\boldsymbol{\omega}$ as one half of the displacement rotor [5] we obtain

$$\Omega_{ij}(\mathbf{r}) \equiv \langle \omega_i^\circ(\mathbf{r}+\mathbf{r}_1)\,\omega_j^\circ(\mathbf{r}_1)\rangle, \quad \boldsymbol{\omega} = {}^1/_2\operatorname{rot}\mathbf{u} \tag{5.1}$$

$$\Omega = {}^1/_4\operatorname{Rot}\mathbf{U}, \qquad \operatorname{div}\Omega = 0 \tag{5.2}$$

Writing the tensor Ω_{ij} in the form

$$\Omega_{ij} = \Omega_{11}\delta_{ij} + (\Omega_{33} - \Omega_{11})n_{ij} \tag{5.3}$$

and using the second equation of (5.2), we find that the components of the angle of rotation correlation tensor are connected by

$$r\Omega_{33}' + 2(\Omega_{33} - \Omega_{11}) = 0, \qquad \Omega_{11} = \frac{1}{2r}\frac{d}{dr}(r^2\Omega_{33}) \tag{5.4}$$

Here the components are written in the tensor notation unlike the stress and strain field components. Equations (5.4) are formally analogous to the Kármán relations connecting the components of the velocity field correlation tensor obtained with the equation of continuity taken into account [6].

6. The derivative of the angle of rotation vector taken with respect to the coordinate, represents a curvature tensor [5]. Its correlation function can be used to estimate the grain bending in a microinhomogeneous medium under a homogeneous macrodeformation. The curvature correlation tensor can be obtained from the relations

$$\Gamma_{ijkl}(\mathbf{r}) \equiv \langle \gamma_{ij}(\mathbf{r}+\mathbf{r}_1)\,\gamma_{kl}(\mathbf{r}_1)\rangle, \qquad \gamma_{ij} \equiv \omega_{i,j} \tag{6.1}$$

$$\Gamma_{ijkl} = -\Omega_{ik,jl} \equiv -\theta_{ikjl} \tag{6.2}$$

The tensor θ_{ikjl} is not symmetric with respect to the interchange of a pair of indices; an interchange of the indices within each pair, however, leaves it unchanged

$$\theta_{ikjl} = \theta_{kijl} = \theta_{iklj} \neq \theta_{ljik}$$

The tensor θ_{ikjl} allows the passage to the standard matrix notation. This is however not true for the tensor Γ_{ijkl} ($\Gamma_{1313} \neq \Gamma_{3131}$), hence below we obtain the connections between the components of θ_{ikjl}.

The passage from θ_{ikjl} to Γ_{ijkl} can be easily executed using the definition (6.2). The tensor θ_{ikjl} is obtained in its explicit form by inserting the expression (5.3) for Ω_{ik} into (6.2)

$$r^2\theta_{ikjl} = r\Omega_1'\delta_{ik}\delta_{jl} + \Omega_2(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) + r(r\Omega_1'' - \Omega_1')\delta_{ik}n_{jl} +$$
$$+ (r\Omega_2' - 2\Omega_2)(\delta_{kl}n_{ij} + \delta_{ij}n_{kl} + \delta_{il}n_{jk} + \delta_{jk}n_{il} + \delta_{jl}n_{ik}) +$$
$$+ (r^2\Omega_2'' - 5r\Omega_2' + 8\Omega_2)n_{ijkl} \tag{6.3}$$

$$\Omega_1 \equiv \Omega_{11}, \qquad \Omega_2 \equiv \Omega_{33} - \Omega_{11} \tag{6.4}$$

Equation (6.3) shows that although the tensors θ_{ikjl} and Γ_{ijkl} have no axial symmetry, they are nevertheless characterized by five independent algebraic components. Introducing the matrix indices we find seven different components of the tensor θ_{ikjl} and two algebraic relations connecting these components

$$\theta_{11} = 2\Omega_2 + r\Omega_1', \qquad \theta_{12} = r\Omega_1', \qquad \theta_{13} = r^2\Omega_1''$$
$$\theta_{31} = r(\Omega_1' + \Omega_2') - 2\Omega_2, \ \theta_{33} = r^2(\Omega_1'' + \Omega_2'') \tag{6.5}$$
$$\theta_{44} = r\Omega_2' - \Omega_2, \ \theta_{66} = \Omega_2,$$
$$\theta_{66} = {}^1/_2(\theta_{11} - \theta_{12}), \ \theta_{31} = \theta_{44} + \theta_{66} - \theta_{12} \tag{6.6}$$

The first equation of (6.6) holds for all systems possessing hexagonal syngony, while the second equation reflects the condition that $\theta_{13} \neq \theta_{31}$. Eliminating from (6.5) the auxilliary functions Ω_1 and Ω_2, we find three differential equations connecting the components of the tensor θ_{ikjl}

$$r\theta_{11}' = \theta_{11} + \theta_{12} + 2\theta_{44}, \quad r\theta_{12}' = \theta_{12} + \theta_{13}, \quad r\theta_{44}' = {}^1/_2(\theta_{33} - \theta_{12}) \tag{6.7}$$

Next, choosing θ_{11} and θ_{12} as the independent functions, we express the remaining components of θ_{ikjl} in the terms of these functions

$$\theta_{13} = (R-1)\theta_{12}, \qquad \theta_{31} = {}^1/_2 R\theta_{11} = 2\theta_{12} \tag{6.8}$$
$$\theta_{33} = (R-1)(R\theta_{11} - \theta_{12}), \qquad 2\theta_{44} = (R-1)\ \theta_{11} - \theta_{12}$$

Using (6.2) we can easily write (6.7) and (6.8) directly for the components of the curvature correlation tensor Γ_{ijkl}. We must however in this case employ the tensor indices, since $\Gamma_{ijkl} \neq \Gamma_{jilk}$.

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ON THE LOWER PORTION OF THE SPECTRUM OF NATURAL AXISYMMETRIC VIBRATIONS OF A THIN ELASTIC SHELL OF REVOLUTION

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Conditions are presented for which the lower part of the spectrum of the membrane problem consists of an infinite series of eigenvalues converging to the lower bound of the continuous spectrum. It is shown that boundary layer theory [1] is applicable to this portion of the spectrum and the first approximation is obtained for the eigenvalues.

The equations of natural axisymmetric vibrations of a thin elastic shell of revolution are [2, 3]:

$$-\frac{d}{ds}\left(\frac{1}{B(s)}\frac{dBu}{ds}\right)-\left(\frac{1-\sigma}{R_1(s)R_2(s)}\right)u+\left(\frac{1}{R_1}+\frac{\sigma}{R_2}\right)\frac{dw}{ds}+\frac{d}{ds}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)w=\lambda u \qquad (0.1)$$

$$-\left(\frac{1}{R_1}+\frac{\sigma}{R_2}\right)\frac{du}{ds}-\left(\frac{\sigma}{R_1}+\frac{1}{R_2}\right)\frac{B'}{B}u+\left(\frac{1}{R_1^2}+\frac{2\sigma}{R_1R_2}+\frac{1}{R_2^2}\right)w+\frac{h^2}{12}\frac{1}{B}\frac{d}{ds}\left(B\frac{d}{ds}\left(B\frac{dw}{ds}\right)\right)=\lambda w$$

Here the parameter s is the arclength of a meridian of the middle surface measured from some fixed point, $B(s)$ is the distance between a variable point on the meridian and the axis of revolution. The projections of displacement of the middle surface point in the directions of the meridian and of the normal to the surface and denoted by $u(s)$ and $w(s)$. For the principal radii of curvature we have

$$R_1^{-1}=-B''(1-(B')^2)^{-1/2}, \qquad R_2^{-1}=(1-(B')^2)^{1/2}B^{-1}$$

The spectral parameter λ is proportional to the square of the vibrations frequency, the small parameter h is the relative shell thickness, and σ is Poisson's ratio. The coefficients of (0.1) are assumed sufficiently smooth.

Let us bound the shell by two parallels $s=s_1$ and $s=s_2$, and let us take the following boundary conditions

$$u(s_1)=u(s_2)=w(s_1)=w(s_2)=w'(s_1)=w'(s_2)=0 \qquad (0.2)$$